# Computabilita'

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## 1 Graphs of computable functions

**Proposition 1** *A function is computable iff its graph* $graph(f) = \{(x, y) : y = f(x)\}$ *is semidecidable.*

A set $V$ of computable functions is called *completely r.e. (decidable)* if the set $\{y : \phi_y \in V\}$ is r.e. (decidable).

Let $b_k$ ($k \in N$) be an enumeration of the finite functions. For every finite function $\theta = \{(x_1, \theta(x_1)), \dots, (x_n, \theta(x_n))\}$, we fix a program $P_\theta$ which computes it. Let $c_\theta$ be the natural number codifying this program $P_\theta$.

**Theorem 2** (Rice, Myhill and Shepherdson 1955) *A set $U$ of computable functions is completely r.e. iff there is an r.e. set $B$ such that*

$$\phi_x \in U \Leftrightarrow (\exists n)(n \in B \wedge b_n \leq \phi_x).$$

**Proof.** The theorem is trivial if either $U = \emptyset$ or $U$ is the set of all computable functions. Then we analise the other cases.

($\Rightarrow$) Let $A = \{y : \phi_y \in U\}$. By hypothesis $A$ is r.e. Since $A$ 'rispetta le funzioni' and $A \neq \emptyset, N$ then the hypotheses of the second and third Rice theorem fail for $A$. This means that

$$f \in U \Leftrightarrow (\exists \theta \in U)(\mathrm{dom}(\theta) \text{ finite } \wedge \theta \leq \phi_x).$$

Then $B = \{n : c_{b_n} \in A\}$.

($\Leftarrow$) Left to the reader. ■

### 1.0.1 Does a function admit a decidable graph?

**Proposition 3** *If a computable function $f$ has a decidable domain, then graph($f$) is decidable.*

**Proof.** Given $(x, y)$, decide whether $x \in dom(f)$; if not, then $(x, y) \notin$ graph$(f)$, else compute $f(x)$. ■

If a computable function does not have a decidable domain, the decidability of its graph depends on the value assumed by the function. The following are two functions having the same domain $K = \{x : \phi_x{\downarrow}x\}$. The first one $f$, defined by

$$f(x) = \text{if } x \in K \text{ then } 1, \tag{1}$$

does not have a decidable graph, while the second one $g$, defined by

$$g(x) = y \text{ iff } \phi_x{\downarrow}x \text{ in exactly } y \text{ steps of computation}, \tag{2}$$

does have.

### 1.0.2 Does a function admit a total computable extension?

By Prop. 3 the question is interesting if the function does not have a decidable domain.

**Proposition 4** *Let $f$ be computable such that dom($f$) is not decidable. If graph($f$) is decidable, then $f$ does not admit a total computable extension.*

**Proof.** Assume that $g$ is a total computable extension of $f$. We can decide *dom*($f$) as follows: Given $x$, we compute $g(x)$ and then we decide whether the pair $(x, g(x))$ is in graph$(f)$. If yes then $x \in dom(f)$, else $x \notin dom(f)$. ■

The function $g$ defined in (2) does not admit a total computable extension.

**Example 5** *The reverse conclusion of Prop. 4 is false. The function $f(x) = \phi_x(x)$ does not admit a total computable extension and has a non-decidable graph. We show that the graph of $f$ is not decidable. Assume the opposite. The the set $I = \{x : \phi_x(x) \geq 2$ or $\phi_x(x)$ undefined$\}$ is decidable (decide the pairs $(x, 0)$ and $(x, 1)$ in the graph of $f$). Then consider the computable function*

$$\phi_{s(x)}(y) = \textit{if } x \in K \textit{ then 1.}$$

*It follows that $K$ is reducible to the complement of $I$. This implies that $I$ is not decidable. Contradiction.*

**Proposition 6** *If a computable function $f$ does not have a decidable domain and admits a total computable extension, then $\{y : \phi_y \not\leq f\}$ is not r.e.*

**Proof.** Define the function (by applying the parameter theorem)

$$\phi_{s(y)}(x) = \text{if } y = x \text{ then } g(x) \text{ else (if } y \in \mathit{dom}(f) \text{ then } f(x) \text{ else } \uparrow).$$

Then,

$$y \in \mathit{dom}(f) \;\Rightarrow\; \phi_{s(y)} = f \;\Rightarrow\; \phi_{s(y)} \leq f$$

and

$$y \notin \mathit{dom}(f) \;\Rightarrow\; \phi_{s(y)}(x) = \text{if } y = x \text{ then } g(x) \text{ else } \uparrow \;\Rightarrow\; \phi_{s(y)} \not\leq f.$$

It follows that the set $\{y : \phi_y \not\leq f\}$ is not r.e. ■

The function $f$ defined in (1) satisfies the hypothesis of Prop. 6.

### 1.0.3 Does a function admit an extension with a decidable graph?

**Proposition 7** *There is a computable function which does not admit extensions with a decidable graph.*

**Proof.** Let $h(x) = \phi_x(x)$. By contraposition assume that there is a computable function $g$ such that $h < g$ and graph$(g)$ is decidable. Define a computable total function $f$ as follows: $f(x) = \text{if } (x, 0) \notin \text{graph}(g) \text{ then } 0 \text{ else } 1$. Since $f$ is computable there is an index $a$ such that $f = \phi_a$. We get a contradiction as follows:

$$f(a) = 0 \;\Rightarrow\; f(a) = \phi_a(a) = g(a) = 0 \;\Rightarrow\; (a, 0) \in \text{graph}(g) \;\Rightarrow\; \text{Contradiction.}$$

$$f(a) = 1 \;\Rightarrow\; f(a) = \phi_a(a) = g(a) = 1 \;\Rightarrow\; (a, 1) \in \text{graph}(g) \;\Rightarrow\; \text{Contradiction.}$$

■

**Remark 8** *Define $f \leq_m g$ if there is a computable total function $s$ such that $(x, y) \in graph(f)$ iff $s(x, y) \in graph(g)$. Then the decidability of graph$(g)$ implies the decidability of graph$(f)$.*

*Let* $h(x) = \phi_x(x)$*. Then* $f \leq_m h$ *for every computable function* $f$*. Define* $\phi_{t(x,y)}(z) =$ *if* $(x, y) \in$ *graph*$(f)$ *then 1 else* $\uparrow$*. Finally, define* $s(x, y) = (t(x, y), 1)$*. If* $(x, y) \in$ *graph*$(f)$ *then* $\phi_{t(x,y)}(z) = 1$ *for all* $z$*, including* $z = t(x, y)$*. This means that* $h(t(x, y)) = \phi_{t(x,y)}(t(x, y)) = 1$*, so that* $s(x, y) \in$ *graph*$(h)$*. If* $(x, y) \notin$ *graph*$(f)$ *then* $\phi_{t(x,y)}$ *is the empty function, so that* $s(x, y) = (t(x, y), 1) \notin$ *graph*$(h)$*. It follows that* $h$ *is the function having the most difficult graph.*

**Problem 9** *To characterize the computable functions which do not admit extensions with a decidable graph.*

**Problem 10** *Does every computable coinfinite function* $f$ *admit a computable extension* $g$ *such that* *dom*$(g) -$ *dom*$(f)$ *is infinite? The answer is no (see Thm. III.4.18 in Odifreddi Vol.I). There exists a maximal set. A set* $A$ *is maximal iff it has only trivial supersets, i.e., if* $B$ *is r.e. and* $A \subseteq B$ *then either* $B$ *is cofinite or* $B - A$ *is finite.*